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INTERNAL AND BOUNDARY CALCULATIONS OF THIN ELASTIC BODIES†

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A method for the separate construction of the main stress–strain state (the internal calculation) and the boundary corrections (the boundary calculations) are discussed in the case of a linear static problem in the theory of shells and plates. It is assumed that the internal calculation is carried out using an iterative process based on the Kirchhoff–Love theory. The boundary calculation involves the construction of antiplane and plane boundary layers, that is, in the initial approximation they reduce to the solution of antiplane and plane problems in the theory of elasticity.

Investigation of the asymptotic behaviour of the boundary corrections shows that near a weakly clamped edge only the correction from the antiplane boundary layer is important and that near a fairly rigidly clamped edge only the correction from the plane boundary layer is important.

The advisability of the use of the shear theory of the bending of plates for investigating boundary elastic phenomena is discussed from the point of view of the results obtained. It is shown that, close to the free edge, its use is justified and is adequate for the method described in the paper both with regard to the numerical results and with regard to the nature of the mathematical apparatus. As a method for investigating boundary elastic phenomena, shear theories lose their meaning close to a fairly rigidly clamped edge since they only enable one to construct the minor part of the correction asymptotically.

1. The elastic properties of a thin isotropic body which is bounded by faces (which are sufficiently extended in two directions) and end faces (which are thin in one of the directions) are discussed. We shall consider the linear static properties of the stress–strain state (SSS) and assume that the external forces on the faces are specified, that is, cases when it is necessary to take account of any clamping of the faces are ignored (these cases are discussed in [1]).

We shall refer to the thin bodies which have been described as shells and, unless otherwise stated, we shall permit these shells to degenerate into plates.

The property of the shell which is expressed by the structural formula

$$(\mathrm{SSS})_{\mathrm{total}} = (\mathrm{SSS})_{\mathrm{int}} + (\mathrm{SSS})_{\mathrm{bound}} \tag{1.1}$$

is the basis of the discussion. In this formula $(\mathrm{SSS})_{\mathrm{total}}$, $(\mathrm{SSS})_{\mathrm{int}}$ and $(\mathrm{SSS})_{\mathrm{bound}}$ represent the total, internal and boundary (generated by an edge or by other stress concentrators) SSS of the shell.

The problem of approximate methods for the internal calculation of the shell, that is, the determination of its $(\mathrm{SSS})_{\mathrm{int}}$ and the boundary calculation, that is, the determination of the $(\mathrm{SSS})_{\mathrm{bound}}$, form the main subject of this treatment. Here both methods are constructed on the basis of the asymptotic integration (for a thin domain) of the three-dimensional linear differential equations of the static theory of isotropic elasticity. Methods which are traditional in the case of such approaches are used here.

1. Integrals with different properties are investigated separately.

2. The asymptotic properties of the integrals are predicted and the assumptions which are adopted are later checked for the correctness of the procedures for solving the boundary-value problems which follow from them.

In the theory of shells, it makes sense to split the overall calculation into internal and external calculations not only from the mathematical point of view but also from a physical point of view, since the practical importance of $(\mathrm{SSS})_{\mathrm{int}}$ and $(\mathrm{SSS})_{\mathrm{bound}}$ is not the same.

The asymptotic approaches used here are compared with engineering approaches. It is shown that the Kirchhoff–Love theory is completely confirmed as an approximate method for investigating $(\mathrm{SSS})_{\mathrm{int}}$. However, the Timoshenko–Reissner theory requires a number of stipulations both as a method for refining the internal calculation and, in particular, as a method for the boundary calculation.

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2. In satisfying the scheme described in Section 1, we shall define $(\mathrm{SSS})_{\mathrm{int}}$ by the formulae

$$
\begin{aligned}
\tau_i &= \left(1+\frac{\alpha_3}{R_j}\right)\sigma_{ii} = \lambda^l(\tau_i^0 + \zeta\lambda^{-l+2p-c}\tau_1^1)\\
\tau_{ij} &= \left(1+\frac{\alpha_3}{R_j}\right)\sigma_{ij} = \lambda^l(\tau_{ij}^0 + \zeta\lambda^{-l+2p-c}\tau_{ij}^1)\\
\tau_{i3} &= \left(1+\frac{\alpha_3}{R_j}\right)\sigma_{i3} = \lambda^p(\tau_{i3}^0 + \zeta\tau_{i3}^1 + \lambda^{-l+2p-c+b}\zeta^2\tau_{i3}^2)\\
\tau_{33} &= \left(1+\frac{\alpha_3}{R_1}\right)\left(1+\frac{\alpha_3}{R_2}\right)\sigma_{33} = \lambda^c(\tau_3^0 + \zeta\tau_3^1 + \lambda^{-l+2p-c+b}\zeta^2\tau_3^2 + \lambda^{-2l+4p-2c+b}\zeta^3\tau_3^3)\\
\upsilon_i &= \lambda^{l-p+b}(\upsilon_i^0 + \zeta\lambda^{-l+2p-c}\upsilon_i^1), \quad \upsilon_3 = \lambda^{l-c+b}(\upsilon_3^0 + \zeta\lambda^{-l+c}\upsilon_3^1)
\end{aligned}
\tag{2.1}
$$

the meaning of which is as follows.

We shall assume that the elastic medium forming the shell is referred to the traditional tri-orthogonal system of coordinates by the equality

$$\mathbf{P} = \mathbf{M}(\alpha_1, \alpha_2) + \alpha_3\mathbf{n} \tag{2.2}$$

where **M** is the vector of the median surface, specified in the lines of curvature and **n** is the unit vector normal to the surface **M**.

The stresses and displacements of the spatial medium, relative to the coordinates (2.2), are implied by σ_{st}, υ_s. The so-called asymmetric stresses are denoted by τ_i, τ_{ij}, τ_{i3}, τ_{33} ($i \neq j = 1, 2$). These must satisfy the face conditions

$$\tau_{r3}|_{\alpha_3=\pm h} = \tau_{r3}^{\pm} \qquad (r = 1,2,3) \tag{2.3}$$

in which $\tau_{r3}^{\pm}$ are known functions specifying the external forces on the faces and h is the half-thickness of the shell.

In addition, the following notation has been used: λ is a large parameter which is defined by the formula

$$\lambda^l = R/h \tag{2.4}$$

R, for a shell, is the characteristic radius of curvature of the median surface and, in the case of a plate, is a certain characteristic dimension of its middle plane, τ_i^k, τ_{ij}^k, τ_{i3}^k, τ_{33}^k, υ_i^k, υ_3^k are functions of the two variables α_1 and α_2, $\zeta = \alpha_3/h$ is a dimensionless normal coordinate, b and p are arbitrary numbers which satisfy the requirements

$$0 \leqslant p < l, \qquad 0 \leqslant b \leqslant l - 2p \tag{2.5}$$

the number c is expressed in terms of l and p by the following formulae: for a shell

$$c = 0 \quad (p \leqslant l/2), \quad c = 2p - l \quad (p \geqslant l/2) \tag{2.6}$$

and, for a plate, for any p

$$c = 2p - l \tag{2.7}$$

and R_j are the principal radii of curvature of the median surface.

It is assumed that the stress and displacement functions (2.1) satisfy the three-dimensional equations of the theory of elasticity and the face conditions (2.3) with a certain asymptotic accuracy (as $\lambda \to \infty$) if τ_i^k, τ_{ij}^k, τ_{i3}^k, τ_{33}^k, υ_i^k, υ_3^k are integrals of a certain two-dimensional system of equations which are equivalent to the equations of the Kirchhoff–Love theory.

The latter assumption can be interpreted more specifically as follows.
We make the scale transformation of the independent variables

$$\alpha_1 = R\lambda^{-p}\xi_1, \quad \alpha_2 = R\lambda^{-p}\xi_2, \quad \alpha_3 = R\lambda^{-l}\zeta \tag{2.8}$$

in the spatial equations of the theory of elasticity which is traditional in the case of asymptotic approaches and we shall consider in the treatment only those integrals of the resulting equations for which, firstly, differentiation of the required quantities with respect to ξ_1 and ξ_2 does not change their asymptotic order and, secondly, all the quantities

$$\frac{1}{E}(\tau_i^k, \tau_{ij}^k, \tau_{i3}^k, \tau_{33}^k) \text{ and } \frac{1}{R}(\upsilon_i^k, \upsilon_3^k)$$

(E is Young's modulus) have the same asymptotic order. The three-dimensional differential equations of the theory of elasticity, written in the coordinates (2.2) with the independent variables ξ_1, ξ_2, ζ and the required functions τ_i^k, τ_{ij}^k, τ_{i3}^k, τ_{33}^k, υ_i^k, υ_3^k are then transformed into equalities in which it is easy to obtain a relative asymptotic estimate for each term. It is determined by factors of the form of λ^θ which appear in the equations of the theory of elasticity as a consequence of the use of formulae (2.1) and (2.8).

If, in each separately taken equality with such properties, terms containing the factor $\lambda^{-\mu}$ are neglected compared with terms containing the factor $\lambda^{-\rho}$ when $\mu \geqslant \rho$, we shall say that the number ρ is a characteristic of their accuracy in the case of the equations obtained in this manner. The hypothesis under discussion then reduces to the following assertion: the three-dimensional equations of the theory of elasticity and the face conditions (2.3) are satisfied with an accuracy characteristic $\rho = 2l - 2p$, if the required quantities, introduced by formulae (2.1), are related to the forces, moments, displacements and angles of rotation (T_i, S_{ij}, G_i, H_{ij}, N_i, υ_i, w, γ_i) of the Kirchhoff–Love theory by the following intermediate formulae

$$\begin{gathered}
\tau_i^0 = \frac{1}{2R}T_i, \quad \tau_{ij}^0 = \frac{1}{2R}S_{ij}, \quad \tau_i^1 = -\lambda^{2l-2p+c-b}\frac{3}{2R^2}G_i \\
\tau_{ij}^1 = \lambda^{2l-2p+c-b}\frac{3}{2R^2}H_{ij}, \quad \tau_{i3}^0 + \frac{1}{3}\lambda^{-l+2p-c+b}\tau_{i3}^2 = -\frac{\lambda^{l-p}}{2R}N_i \\
\tau_{i3}^1 + \lambda^{-l+2p-c+b}\tau_{i3}^2 = \frac{\lambda^{-p}}{2}(\tau_{i3}^+ + \tau_{i3}^-)
\end{gathered} \tag{2.9}$$

$$\begin{gathered}
\tau_{i3}^1 = \frac{\lambda^{-p}}{2}(\tau_{i3}^+ + \tau_{i3}^-) \\
\upsilon_i^0 = \lambda^{-l+p-b}u_i, \quad \upsilon_3^0 = -\lambda^{-l+c-b}w \\
\upsilon_i^1 = -R\lambda^{-l-p+c-b}\gamma_i, \quad \upsilon_3^1 = -\lambda^{-c}\frac{\nu}{2E}(T_1 + T_2) \\
(i \neq j = 1, 2)
\end{gathered}$$

and if the two-dimensional quantities which have been enumerated above on the right-hand sides of Eqs (2.9) satisfy the two-dimensional equations of the Kirchhoff–Love theory.

The property of $(\text{SSS})_{\text{int}}$, expressed by formulae (2.1), can be directly verified by substituting (2.1) and (2.8) into the three-dimensional equations of the theory of elasticity and the face conditions (2.3) using the above-mentioned rule for neglecting small terms. The corresponding long, but elementary, calculations are presented in [2], from where the notation has also been borrowed.

3. The asymptotic form of a certain class of stress–strain states of a thin spatial elastic body, on the faces of which the external forces are specified, is given by formula (2.1). Such stress–strain states can be identified with $(\text{SSS})_{\text{int}}$ in the structural formula (1.1) with an accuracy characteristic $\rho = 2l - 2p$ and, using the intermediate formulae (2.9), it is possible to construct $(\text{SSS})_{\text{int}}$ using the Kirchhoff–Love theory.

The numbers l, p and b in (2.1) have the following physical meaning.

The number l, for fixed h, R, λ, is defined by formula (2.4). A fundamentally important asymptotic property of the stress–strain state under discussion is specified by the ratio $p/l = t$. It follows from the first two formulae of the scale transformation (2.8), after they have been transformed to the form

$$\alpha_1 = \left(\frac{h}{R}\right)^t R\xi_1, \quad \alpha_2 = \left(\frac{h}{R}\right)^t R\xi_2$$

that t has the meaning of an index of the variability of a given SSS with respect to the variables α_1 and α_2, that is, in the median surface of the shell.

Relations (2.6) and (2.7) hold for the number c. The first of these are connected with the generally known fact that, if a shell does not degenerate into a plate, then two kinds of $(\mathrm{SSS})_{\mathrm{int}}$ exist in it. When $t < 1/2$, the forces and, when $t > 1/2$, the moments in them are asymptotically predominant. On account of this, plates are sometimes treated here not as the simplest shells but as thin elastic bodies with special properties.

The number b in formulae (2.1) appears because, in a shell which does not degenerate into a plate, $(\mathrm{SSS})_{\mathrm{int}}$ can have two types of asymptotic forms. They have been called [2] normal asymptotic forms (when $b = 0$) and special asymptotic forms (when $0 < b < l - 2p$).

From a physical point of view, the asymptotic form of $(\mathrm{SSS})_{\mathrm{int}}$ become special when the deformation of the middle surface is a pseudobending [3], that is, a change in shape which is close in a certain sense to that which is called an infinitesimal bending in the theory of surfaces. In practical problems, pseudobending of the median surface is a not uncommon phenomenon. However here, in order to avoid multivariant arguments, we shall always put $b = 0$, that is, we shall assume that the asymptotic form of the $(\mathrm{SSS})_{\mathrm{int}}$ is normal.

4. The approximate construction of $(\mathrm{SSS})_{\mathrm{int}}$ can also be carried out for a higher value of the accuracy characteristic, that is, when $\rho > 2l - 2p$, using asymptotic methods. In order to do this, it is necessary, in particular, to make formulae (2.1), which specify the properties of the $(\mathrm{SSS})_{\mathrm{int}}$, more complicated. For instance, when $\rho > 4l - 4p$, it will be necessary, in each of the circular brackets on the right-hand sides of (2.1), to increase by unity the degree of the polynomials in ζ appearing there. The required system of two-dimensional equations becomes correspondingly more complex and there is an increase in its order. Such systems have been obtained for shells [4] and also for plate bending and plate extension (compression) [5, 6]. It has been found that, while the asymptotic theory when $\rho = 2l - 2p$ can be considered as an analogue of the Kirchhoff–Love theory, when $\rho = 4l - 4p$, one can speak of an asymptotic analogue of shear theory (Timoshenko–Reissner shear theory is to be understood henceforth). However, the analogy is far less complete in the second case than in the first. This has been emphasized in [4–6] and will be still more specifically discussed here.

For now, we note a feature of the analogy: both in the asymptotic theory $\rho > 2l - 2p$ and in the shear theories, there is an increase in the order of the two-dimensional differential equations. This means that the transition from $\rho = 2l - 2p$ to $\rho = 4l - 4p$ corresponds not only to a refinement of the solutions given by the Kirchhoff–Love theory, but also to the appearance of "additional" solutions, and they have a variability index $t = 1$ in both the asymptotic theory $\rho = 4l - 4p$ and shear theory constructed on the basis of physical hypotheses. This imparts the features of an analogy with $(\mathrm{SSS})_{\mathrm{bound}}$ to the "additional" solutions, but is also evidence of their mathematical insufficiency since the asymptotic method is based on the assumption that $t < 1$. This issue will be further and more specifically discussed here.

5. The asymptotic properties of $(\mathrm{SSS})_{\mathrm{bound}}$ in the structural formula (1.1) are determined in the following manner.

We shall assume that a shell has single closed edge and choose the tri-orthogonal system of coordinates (2.2) such that this edge coincides with the coordinate surface $\alpha_1 = 0$.

We shall make the following substitutions in the equations of the theory of elasticity for the stresses σ_{ik} and the displacements υ_k, and, also, for the independent variables α_k.

$$S_{ii} = \left(1 + \frac{\alpha_3}{R_j}\right)\frac{\sigma_{ii}}{E}, \quad S_{ij} = \left(1 + \frac{\alpha_3}{R_i}\right)\frac{\sigma_{ij}}{E}$$

$$S_{i3}=\left(1+\frac{\alpha_3}{R_j}\right)\frac{\sigma_{i3}}{E},\quad S_{33}=\left(1+\frac{\alpha_3}{R_1}\right)\left(1+\frac{\alpha_3}{R_2}\right)\frac{\sigma_{33}}{E}$$
$$V_k=h^{-1}\upsilon_k,\quad \alpha_1=R\lambda^{-l}\eta_1,\quad \alpha_2=R\lambda^{-\pi}\eta_2,\quad \alpha_3=R\lambda^{-l}\zeta \qquad (5.1)$$
$$(i\neq j=1,2;\quad k=1,2,3)$$

In these formulae it is assumed that ζ, λ, l have the same meaning as in (2.1). We mean by $\pi \leqslant p$, a number related by the formula $\tau = \pi/l$ to the index of variability with respect to the variable α_2. This means that τ_1 is a so-called partial variability index of $(\mathrm{SSS})_{\mathrm{bound}}$, that is, it is a characteristic of the variability along the edge of the shell. It is usually known from the conditions of the problem and is related to the general variability index t by the inequality $\tau_1 \leqslant p$.

By $(\mathrm{SSS})_{\mathrm{bound}}$, it is necessary to understand that set of quantities S_{ik}, V_k which corresponds to those integrals of the transformed system of equations of the theory of elasticity in which differentiation with respect to η_1, η_2, ζ does not change the asymptotic order of the required functions.

Apart from this, we require that S_{ik}, V_k should be of the form $O(\lambda^{-\mu})$ with a single μ for all of these quantities. Then, the differential equations of the theory of elasticity for $(\mathrm{SSS})_{\mathrm{bound}}$ will possess the same properties as were indicated in Section 2 in the case of the equations defining $(\mathrm{SSS})_{\mathrm{int}}$. In these equations, it is easy to determine the relative asymptotic of each term in any equation which is taken separately. Also, as was shown in [2, 7, 8], $(\mathrm{SSS})_{\mathrm{bound}}$ is defined with a characteristic accuracy $\rho = l - \pi$ by two systems of equations.

System *a*

$$\frac{1}{A_1}\frac{\partial S_{12}}{\partial \eta_1}+\frac{\partial S_{23}}{\partial \zeta}=0$$
$$\frac{1}{A_1}\frac{\partial V_2}{\partial \eta_1}-2(1+\nu)S_{12}=0,\quad \frac{\partial S_2}{\partial \zeta}-2(1+\nu)S_{23}=0 \qquad (5.2)$$

System *b*

$$\frac{1}{A_1}\frac{\partial S_{11}}{\partial \eta_1}+\frac{\partial S_{13}}{\partial \zeta}=0,\quad \frac{1}{A_1}\frac{\partial S_{13}}{\partial \eta_1}+\frac{\partial S_{33}}{\partial \zeta}=0$$
$$\frac{1}{A_1}\frac{\partial V_1}{\partial \eta_1}-[S_{11}-\nu(S_{22}+S_{33})]=0,\quad -[S_{22}-\nu(S_{11}+S_{33})]=0 \qquad (5.3)$$
$$\frac{\partial V_3}{\partial \zeta}-[S_{33}-\nu(S_{11}+S_{22})]=0,\quad \frac{\partial V_1}{\partial \zeta}+\frac{1}{A_1}\frac{\partial V_3}{\partial \eta_1}-2(1+\nu)S_{13}=0$$

(in these equations, A_1 is the coefficient of the first quadratic form of the median surface).

System (5.2) is closed with respect to the quantities $P = (S_{12}, S_{23}; V_2)$ while system (5.3) is closed with respect to the quantities $Q = (S_{11}, S_{22}, S_{33}, S_{13}; V_1, V_3)$. The two-dimensional equations (5.2) of the so-called antiplane problem of the theory of elasticity are obtained for P while the equalities (5.3) form the equations of the plane problem of the theory of elasticity for Q (in both cases $\eta_1 = \int_0^{\alpha_1} A_1 d\alpha_1$ and ζ are the independent variables).

The initial approximation which has been described above can be refined by the method of iterations and, apart from the principal stresses and displacements, the secondary stresses and displacements can be constructed asymptotically. Their asymptotic form is expressed by the relations derived in [2]

$$Q'(a)=O[\lambda^{-l+\pi}P(a)],\qquad P'(b)=O[\lambda^{-l+\pi}Q(b)] \qquad (5.4)$$

Quantities which are asymptotically secondary for an SSS of a given form will henceforth be given a prime.

Together, P and Q encompass all the required quantities of a certain $(\mathrm{SSS})_{\mathrm{bound}}$. In cases when the asymptotically principal part of the stresses and displacements is determined by system *a*, the $(\mathrm{SSS})_{\mathrm{bound}}$ is called an antiplane boundary layer and, when the principal part is determined by system *b*, we shall speak of a plane boundary layer. The corresponding stresses and displacements will be labelled with the additional superscripts *a* and *b*.

It is seen from formula (5.4) that $(\mathrm{SSS})^a_{\text{bound}}$ and $(\mathrm{SSS})^b_{\text{bound}}$ are directly opposite to one another as regards their asymptotic properties: the principal stresses and displacements of the antiplane boundary layer (a) are asymptotically secondary for the plane layer (b) and vice versa.

The properties of an $(\mathrm{SSS})_{\text{bound}}$ described above enable us to present the structural formula in greater detail and to write it as

$$(\mathrm{SSS})_{\text{total}} = (\mathrm{SSS})_{\text{int}} + \lambda^{\alpha}\,(\mathrm{SSS})^a_{\text{bound}} + (\mathrm{SSS})^b_{\text{bound}} \tag{5.5}$$

In this equality the exponents α and β are assumed to be still undetermined. It is explained below that it is necessary to assign some value to them, depending on the nature of the clamping of the edge of the shell.

6. We shall describe the arguments leading to the determination of the exponents α and β in the structural formula (5.5) using an example, when the edge of the shell coincides with the coordinate surface $\alpha_1 = 0$ and the conditions

$$\tau_1 = 0, \qquad \tau_{12} = 0, \qquad \tau_{13} = 0 \tag{6.1}$$

are specified on it, which denote the absence of edge clamping (it is assumed that the SSS being discussed is caused by forces which are distributed over the shell faces and there are no edge actions).

In (6.1), the asymmetric stresses, the meaning of which is determined by the first three relations (2.1), are denoted by τ_1, τ_{12}, τ_{13}.

We use the structural formula (5.5), expand $(\mathrm{SSS})_{\text{int}}$ using formulae (2.1), and $(\mathrm{SSS})_{\text{bound}}$ using relations (5.1) and (5.4) and rewrite (6.1) as follows:

$$\begin{gathered}
\lambda^{l}(\tau_1^0 + \zeta\lambda^{-l+2p-c}\tau_1^1) + \lambda^{-l+p+\alpha}ES_{11}'(a) + \lambda^{\beta}ES_{11}(b) = 0 \\
\lambda^{l}(\tau_{12}^0 + \zeta\lambda^{-l+2p-c}\tau_{12}^1) + \lambda^{\alpha}ES_{12}(a) + \lambda^{-l+p+\beta}ES_{12}'(b) = 0 \quad (\alpha_1 = 0) \\
\lambda^{p}(\tau_{13}^0 + \zeta\tau_{13}^1 + \lambda^{-l+2p-c}\zeta^2\tau_{13}^2) + \lambda^{-l+p+\alpha}ES_{13}'(a) + \lambda^{\beta}ES_{13}(b) = 0
\end{gathered} \tag{6.2}$$

Henceforth, in terms referring to $(\mathrm{SSS})_{\text{bound}}$, the partial variability index π is replaced by the general index p. It is easy to trace that this has no effect on the final conclusions since $\pi \leqslant p$.

It follows from (2.4) and (2.6) that, in (6.2), the factors λ^l, λ^p, λ^c for a shell which does not degenerate into a plate can be expressed as

$$\lambda^l = \frac{R}{h}, \quad \lambda^p = \left(\frac{R}{h}\right)^{p/l} = \left(\frac{R}{h}\right)^t; \quad \lambda^c = \begin{cases} (R/h)^0, & p \leqslant l/2 \\ (R/h)^{2t-1}, & p \geqslant l/2 \end{cases}$$

and it may be assumed that the exponents of λ appearing here are fixed if the half thickness h, the characteristic radius R and the variability index t of the required SSS are known. It is necessary to express the exponents α and β in (6.2) in terms of l, p and c so that contradictions do not arise, the meaning of which is revealed below. In the case under consideration (a free edge) they have to be specified by the formulae

$$\alpha = 2p - c, \qquad \beta = p \tag{6.3}$$

Then, if the first terms in Eqs (6.2) are expressed in terms of the forces and moments of the Kirchhoff–Love theory using (2.9), we obtain

$$\begin{gathered}
\lambda^{p}[\lambda^{-l+2p-c}ES_{11}'(a) + ES_{11}(b)] = r_{11} = -\frac{T_1}{2h} + \zeta\frac{3G_1}{2h^2} \\
\lambda^{2p-c}[ES_{12}(a) + \lambda^{-l+c}ES_{12}'(b)] = r_{12} = -\frac{S_{12}}{2h} - \lambda^{2p-c}\zeta\tau_{12}^1 \quad (\alpha_1 = 0) \\
\lambda^{p}[\lambda^{-l+2p-c}ES_{13}'(a) + ES_{13}(b)] = r_{13} = -\lambda^{p}[\tau_{13}^0 + \zeta\tau_{13}^1 + \lambda^{-l+2p-c}\zeta^2\tau_{13}^2]
\end{gathered} \tag{6.4}$$

(Here, of course, S_{12} does not have the same meaning as $S_{12}(a)$ and $S_{12}(b)$.)

We shall assume for the present that the quantities r_{11}, r_{12}, r_{13} in (6.4) are known and that equalities (6.4) are the boundary conditions for the $(SSS)_{bound}$ boundary layer. According to the iterative scheme for the integration of the three-dimensional equations of the theory of elasticity [9] it is necessary to add to them the conditions for the modified Saint Venant principle to be applicable, that is, to require that $(SSS)_{bound}$ should decay "in the main" (apart from quantities which are assumed to be negligibly small in the initial approximation of the iterative procedure which is adopted). Four boundary conditions for $(SSS)_{int}$ follow from this. They have been discussed in detail in [2, 10] and, in the initial approximation, are identical with the well-known boundary conditions of the Kirchhoff–Love theory.

$$T_1 = S_{12} = G_1 = N_1 + \partial H / \partial s = 0 \tag{6.5}$$

Hence, an important feature of the asymptotic process of the integration of the three-dimensional equations of the theory of elasticity for thin bodies (when the faces are not clamped) is revealed in the case under consideration (a free edge). In it, for an accuracy characteristic $\rho = 2l - 2p$, the problem of the approximate construction of $(SSS)_{int}$ separates into an independent treatment which is equal to the calculation of a shell using the Kirchhoff–Love theory, taking account of the four boundary conditions adopted in it. Generally speaking, this assertion also holds for other edge clamping conditions (exceptions are possible but they do not involve important practical cases). Consequently, it can be asserted that the Kirchhoff–Love theory is a mathematically based approximate method (when $\rho = 2l - 2p$) for the internal calculation of shells.

We now return to the case when the edge is free and assume that the boundary-value problem for constructing $(SSS)_{int}$ has already been solved, that is, we shall assume that all the quantities τ_i^k, τ_{ij}^k, τ_{i3}^k occurring in (6.2) relating to it are known. They are assumed (Section 2) to be functions of the same asymptotic order. However, for certain values of α_1 and α_2, these or others of them may pass through zero. In particular, it follows from the boundary conditions (6.5) and formula (2.9) that $\tau_1^0 = \tau_1^1 = \tau_{21}^0 = 0$ when $\alpha_1 = 0$. The equalities

$$\begin{aligned} &\lambda^{-l+2p-c} ES'_{11}(a) + ES_{11}(b) = 0 \\ &ES_{12}(a) + \lambda^{-l+c} ES'_{12}(b) = -\zeta\tau_{12}^1 \quad (\alpha_1 = 0) \\ &\lambda^{-l+2p-c} ES'_{13}(a) + ES_{13}(b) = -\tau_{13}^0 - \zeta\tau_{13}^1 - \lambda^{-l+2p-c}\zeta^2\tau_{13}^2 \end{aligned} \tag{6.6}$$

are therefore valid, and these equalities can be treated as boundary conditions for the antiplane and plane boundary layers.

The non-contradictory nature of the boundary conditions (6.6) and, consequently, the validity of the choice of α and β using formulae (6.3) arises from the following considerations. By virtue of (2.5) and (2.6), the inequalities

$$-l + 2p - c \leqslant 0, \qquad -l + c < 0 \tag{6.7}$$

are valid for a shell (and a plate).

It follows from these that relations (6.6) enable one to take the limit as $\lambda \to \infty$ since there are no positive powers of λ in them. At the same time, the limiting boundary equalities (6.6), generally speaking, subdivide into two groups. The second equality (6.6), when $\lambda \to \infty$, is transformed into the boundary condition for an antiplane boundary layer while the first and third equalities of (6.6) define two boundary conditions of a plane boundary layer. The case when it is necessary to take the equality sign in the first relation of (6.7) is an exception (in a plate, this occurs for any p and, in a shell, when $p \geqslant l/2$). It is then necessary to take account of the corrections due to the antiplane boundary layer (the terms marked with primes) in the boundary conditions for the plane boundary layer. In all cases, the number of limiting conditions (6.6) corresponds to the order of the differential equations for which they must be set out. This is a first indication of the consistency of formula (6.3) for the numbers α and β.

We further note that the edge values of the quantities τ_{12}^1, τ_{13}^0, τ_{13}^1, τ_{13}^2 do not belong to those which, when $\alpha_1 = 0$, must vanish by virtue of the boundary conditions of the Kirchhoff–Love theory, and, to be specific, we accept that these quantities are of the order of λ^0 for a selected intensity of the external forces. The limiting boundary-value problems for the antiplane and plane boundary layers will then, in the general case, be inhomogeneous, and it becomes impossible for either $(SSS)_{bound}^a$ or $(SSS)_{bound}^b$ to be identically equal to zero. This is the second indication of the consistency of formulae (6.3) for α and β.

Remark. In special cases, the boundary values of the quantities which have been enumerated can become identically zero. This means that formulae (6.3) for α and β must be changed in the case of such problems.

It can be shown that, if one ignores the special cases which have been mentioned, the indications of consistency in the choice of the values of α and β given here will only be observed when formulae (6.3) are used.

The corresponding arguments are simple in principle, but require a thorough inspection of many versions. We shall not dwell on them here.

7. In the general case, a pair of formulae for the exponents α and β, a group of four boundary conditions for the Kirchhoff–Love theory and the limiting conditions for the $(SSS)_{bound}$ correspond to each group of three conditions in the theory of elasticity. For instance [2, 10, 11], in the case of a hinged, supported edge,

$$\sigma_{11} = \upsilon_2 = \upsilon_3 = 0 \tag{7.1}$$

and, for a clamped edge

$$\upsilon_1 = \upsilon_2 = \upsilon_3 = 0 \tag{7.2}$$

the numbers α and β have to be specified by the formulae

$$\alpha = p, \qquad \beta = l \tag{7.3}$$

In the case of relations (7.2), they lead to the boundary conditions

$$u_1 = u_2 = w = \gamma_1 = 0$$

and the limiting boundary conditions

$$RV_1(b) + \lambda^{-l+2p-c}RV_1'(a) = 0, \qquad V_2(a) + V_2'(b) = 0 \tag{7.4}$$

$$RV_3(b) + \lambda^{-l+2p-c}RV_3'(a) = -\zeta\upsilon_3^1 - \zeta^2\lambda^{-l+2p-c}\upsilon_3^2 \tag{7.5}$$

and, in the case of relations (7.1), to the boundary conditions

$$T_1 = u_2 = w = G_1 = 0$$

and the limiting boundary conditions

$$ES_{11}(b) = 0,\ RV_3(b) + \lambda^{-l+2p-c}RV_3'(a) = -\varsigma V_3^1 - \varsigma^2\lambda^{-l+2p-c}V_3^2,\ RV_2(a) + RV_2(b) = 0 \tag{7.6}$$

This follows from the same considerations as in Section 6. Without repeating them, we note solely that, here, formula (2.1) for υ_3 had to be refined and taken in the form

$$\upsilon_3 = \lambda^{l-c}(\upsilon_3^0 + \zeta\lambda^{-l+c}\upsilon_3^1 + \zeta^2\lambda^{-2l+2p}\upsilon_3^2) \tag{7.7}$$

A term with a factor ζ^2 was introduced into this equality on the basis of general asymptotic reasoning. It is necessary to take account of this because, when $c = 2p - l$, it is commensurate with the term with the factor ζ.

Remarks. 1. The need to refine (7.7) means that not only the hypothesis concerning the non-extensibility of a normal element but also the supposition regarding a linear distribution of the bending υ_3 throughout the thickness are too crude for the investigation of boundary phenomena in these bodies.

2. Strictly speaking, the terms in the limiting boundary conditions (7.4) and (7.5) with the factor $\lambda^{-l+2p-c}$ only have to be retained when $c = 2p - l$.

3. The inclusion of a square factor in formula (7.7) does not mean that it is necessary to give up using the Kirchhoff–Love theory for the internal calculation of thin elastic bodies. The value of υ_3^2 can be found by the method of iterations [5, 6]. The formulae (which is common to plates and shells)

$$\upsilon_3^2 = \nu(\tau_1^1 + \tau_2^1)/2$$

holds for this in which τ_1^1, τ_2^1 can be determined from a calculation using the Kirchhoff–Love theory using the intermediate formulae (2.9).

4. If one is dealing with the bending of a plate, then it will follow from the last formulae of (2.1) and the boundary condition $\upsilon_3^0 = 0$ that only the "refining" term will remain on the right-hand side of formula (7.7) and on the right-hand side of the limiting equation (7.5).

8. We shall now discuss so-called shear theories from the point of view of the results which have been obtained and, for brevity, we shall start out from the version of the shear theory of the bending of plates described in [12]. In the notation adopted there, the initial relations of this theory are written as

$$\begin{aligned}
M_{rr} &= -D\left[\upsilon_{,rr} + \nu\left(\frac{\upsilon_{,r}}{r} + \frac{\upsilon_{,\theta\theta}}{r^2}\right)\right] + \frac{2}{\lambda^2}\left(\frac{\chi_{,\theta r}}{r} - \frac{\chi_{,\theta}}{r^2}\right)\\
M_{\theta\theta} &= -D\left[\frac{\upsilon_{,r}}{r} + \frac{\upsilon_{,\theta\theta}}{r^2} + \nu\upsilon_{,rr}\right] - \frac{2}{\lambda^2}\left(\frac{\chi_{,\theta r}}{r} - \frac{\chi_{,\theta}}{r^2}\right)\\
M_{r\theta} &= -(1-\nu)D\left[\frac{\upsilon_{,\theta r}}{r} - \frac{\upsilon_{,\theta}}{r^2}\right] + \frac{2}{\lambda^2}\left(\frac{\chi_{,r}}{r} + \frac{\chi_{,\theta\theta}}{r^2}\right) - \chi\\
&\left(D = \frac{Eh^3}{12(1-\nu^2)}, \quad \lambda^2 = \frac{20(1+\nu)}{Eh^3}G\right)
\end{aligned} \tag{8.1}$$

Here, h is the overall thickness, υ and χ are required functions, of which the first is related to the flexure w by the formula

$$\upsilon = w + BD\nabla^2 w \quad (B = 6/(5Gh)) \tag{8.2}$$

∇^2 is the Laplacian operator and r and θ are polar coordinates.

When there are no external forces, the equations

$$D\nabla^2\nabla^2 w = 0, \qquad \nabla^2\chi - \lambda^2\chi = 0 \tag{8.3}$$

hold for w and χ.

It may therefore be assumed that one is concerned with a combination of two approximate methods: one of them is obtained when $w \not\equiv 0$, $\chi \equiv 0$, and the other when $w \not\equiv 0$, $\chi \equiv 0$.

The first method obviously has to be considered as a refined approach (with respect to the Kirchhoff–Love theory) to the construction of $(\mathrm{SSS})_{\mathrm{int}}$. We therefore put $\chi = 0$ in (8.1) and compare the resulting formulae with the corresponding relations which follow from the asymptotic method in the case of an accuracy characteristic $\rho = 4l - 4q$, that is, which follow from the asymptotic analogue of the shear theory of the bending of plates [5, 6].

In the notation used in [12], the relations in [5, 6] are written as

$$\begin{aligned}
M_{rr} &= -D\left[w_{,rr} + \nu\left(\frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2}\right)\right] -\\
&- Dh^2\left\{\frac{8-3\nu}{10(1-\nu)}(\nabla^2 w)_{,rr} + \frac{\nu(4+\nu)}{10(1-\nu)}\left[\frac{(\nabla^2 w)_{,r}}{r} + \frac{(\nabla^2 w)_{,\theta\theta}}{r^2}\right]\right\}\\
M_{\theta\theta} &= -D\left[\frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2} + \nu w_{,rr}\right] -\\
&- Dh^2\left\{\frac{8-3\nu}{10(1-\nu)}\left[\frac{(\nabla^2 w)_{,r}}{r} + \frac{(\nabla^2 w)_{,\theta\theta}}{r^2}\right] + \frac{\nu(4+\nu)}{10(1-\nu)}(\nabla^2 w)_{,rr}\right\}\\
M_{r\theta} &= -(1-\nu)D\left[\frac{w_{,\theta r}}{r} - \frac{w_{,\theta}}{r^2}\right] - Dh^2\frac{8+\nu}{10}\left[\frac{(\nabla^2 w)_{,\theta r}}{r} - \frac{(\nabla^2 w)_{,\theta}}{r^2}\right]
\end{aligned} \tag{8.4}$$

(h is the half-thickness of the plate).

Comparison of (8.4) and (8.1) reveals a discrepancy (which is numerically small but does not vanish when $h \to 0$) in all terms in which formulae (8.1) differ from the formulae of the Kirchhoff–Love theory when $\chi \equiv 0$. This is explained by the fact that one of the two following techniques is used in the construction of shear theories.

1. The formal extension of the variational principles of the spatial theory of elasticity to the two-dimensional theory of shells. It is assumed, for example, that the forces perform work in the displacements of the median surface and that the moments perform work in the angles of rotation. However, it has been shown in [2, 13] that the error in such an assumption is of the same order of magnitude as the error in the hypothesis that a normal is preserved.

2. The use of a three-dimensional formulation of variational principles in conjunction with the hypothesis of a linear distribution of the main stresses and strains throughout the thickness of the shell. However, asymptotic analysis shows that, with respect to the order of the errors, this is also equivalent to the hypothesis of the preservation of the normal.

Hence, in carrying out an internal calculation, shear theory leads to amendments to the Kirchhoff–Love theory which require a certain correction. It turns out to be insignificant in the case of the theory of the bending of plates. However, numerical results are presented in [4] which show that the correction may turn out to be substantial in the theory of shells.

The method corresponding to $w \equiv 0$, $\chi \not\equiv 0$, in the shear theory of the bending of plates [12] has to be treated as an approximate technique for constructing $(SSS)_{bound}$. However, comparison with the results of the asymptotic integration of the three-dimensional equations of the theory of elasticity does not confirm this in the general case, not only quantitatively but not even qualitatively.

9. In the overwhelming majority of important practical cases the boundary calculation of a shell, that is, the construction of $(SSS)_{bound}$, can be carried out as a second and not always obligatory stage, assuming that $(SSS)_{int}$ has already been constructed. In particular, in the case of a free edge, the boundary calculation of the bending of a plate reduces in the initial approximation to the integration of Eqs (5.2) of the antiplane problem in the theory of elasticity taking into account the homogeneous face conditions, the condition of modified Saint Venant decay and the secondary boundary condition (6.6). In the latter condition, the term with a small factor λ^{-l+c}, which in the case of a plate is equal to λ^{-2l+2p}, can be discarded, that is, the boundary condition can be reduced to the form

$$ES_{12}(a) = -\zeta\tau^1_{12} \quad (\eta_1 = 0) \tag{9.1}$$

In this equality it has to be assumed that the boundary value τ^1_{12} is known. It is determined using formulae (2.9) by an internal calculation using the Kirchhoff–Love theory, which has been carried out earlier. Since the boundary calculation is carried out approximately and intended for constructing exponentially decaying solutions, it is possible to put $A_1^{-1}\partial/\partial\eta_1 = \partial/\partial\eta_1$ in system (5.2) and to change to an antiplane problem, which is solved in Cartesian coordinates for the half strip $\{\eta_1 \geqslant 0, -1 \leqslant \zeta \leqslant +1\}$.

We further note that the quantity $\tau_{12}|_{\eta_1=0}$ in the boundary condition (9.1) is a function of a single variable η_2, and derivatives with respect to this variable do not occur in Eq. (5.2). Consequently, (9.1) can be replaced by the boundary condition

$$ES_{12}(a) = -\zeta \quad (\eta_1 = 0) \tag{9.2}$$

and, in the final analysis, it is possible to obtain a standard problem to the solution of which an approximate boundary calculation of a plate in the region of the free edge always reduces (it is assumed that the solution of the standard problem must be multiplied by the boundary value of the quantity τ^1_{12}).

Remark. In [2, 10], standard problems are also derived for other conditions for the clamping of the edge.

Hence, in cases, when it is necessary to investigate not only the internal but also the boundary zones of a shell, there is no need to construct two-dimensional theories of a higher order than the Kirchhoff–Love theory (shear theories, for example). The splitting of the overall calculation into an internal calculation, for which the Kirchhoff–Love theory is mathematically well founded, and a boundary calculation, which approximately reduces to solving antiplane and plane problems in the theory of elasticity, appears to be more natural both from the mathematical and the physical points of view.

The method corresponding to $w \equiv 0, \chi \not\equiv 0$ in the shear theory of the bending of plates [12] obviously has to be treated as an approximate technique for constructing $(SSS)_{bound}$. However, comparison with results of the asymptotic integration of the three-dimensional equations of the theory of elasticity does not confirm such an assumption in the general case. This will be further discussed below, but, for the present, we note that, when the above-mentioned method is used, it is necessary to retain just the terms with the function χ in formulae (8.1) and to assume that χ satisfies the second equation of (8.3). Allowing for this, it is easy to find the asymptotic form of the corresponding SSS. These are identical with the first asymptotic form (5.4) which is distinctive for $(SSS)^{a}_{bound}$ but directly opposite to the second asymptotic form (5.4) which is characteristic of $(SSS)^{b}_{bound}$.

The structural formula (5.5) specifies the asymptotic behaviour of the stresses which arise close to the edge of the shell. To fix our ideas, if we assume τ^{k}_{i}, τ^{k}_{ij} in formulae (2.1) have the form $O(\lambda^{0})$, then the contributions of the terms $\lambda^{\alpha}(SSS)^{a}_{bound}$ and $\lambda^{\beta}(SSS)^{b}_{bound}$ to the boundary value of $(SSS)_{total}$ will be asymptotically negligible (with respect to the stresses), while the inequalities $\alpha < l$ and $\beta < 1$ are satisfied, respectively.

The following conclusions ensue from this. According to formulae (6.3) for α and β close to the free edge of a plate (when $c = 2p - l$), the antiplane boundary layer, which is taken separately, approximately determines the boundary correction to the stresses. If it is a question of a shell, which has not degenerated, with a free edge in the case of a small value of π, the index of variability of the external action, that is, when $c = 0$, $\pi < l/2$, then the boundary correction will, in general, be asymptotically unimportant. Formulae (7.3) hold for α and β when the edge is clamped. It follows from these points that, in a shell (in particular, when it degenerates into a plate), the boundary correction will not be asymptotically negligible in this case for any values of the index of variability π of the boundary data. It is approximately determined by the plane boundary layer which has been taken separately.

Remark. These general considerations agree with the results in [14]. The effect of the application of axially symmetric static and kinematic actions at the edge of a closed circular cylindrical shell is considered basing on the three-dimensional theory of elasticity in this paper, that is, a situation is discussed in which an antiplane boundary layer is excluded by virtue of the axial symmetry and the boundary corrections, according to the assertions which have been formulated here, do not asymptotically disappear when and only when the components of the displacements are specified at the edge. This is confirmed by the results in [14].

10. We now compare, from a formal mathematical point of view, boundary calculations in the theory of the bending of plates carried out using the asymptotic method described in Section 9 and the method which corresponds to $w \equiv 0, \chi \not\equiv 0$ and follows from the shear theory [12].

In the polar coordinates r and θ, the latter method leads to integration of the equation

$$\frac{\partial^2\chi}{\partial r^2} + \left[\frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right] - \frac{10}{h^2}\chi = 0 \tag{10.1}$$

All of its solutions when $h \to 0$ have a large variability with respect to r. Hence, if the boundary $r = r_0$ is considered to be not too close to the pole and it is assumed that the variability of the required solution with respect to θ is not too large, then the terms in the square brackets in (10.1) can be neglected. Their presence is unjustified at the degree of approximation at which Eq. (10.1) was derived.

Within the framework of the approaches which have been described here in formulae (6.3), it is necessary, in the case of a plate, to interpret c in accordance with formula (2.7). We shall put $\alpha = l > \beta = p$. This means that the boundary calculation approximately reduces to the construction of the decaying solution of Eqs (5.2) of the antiplane problem while satisfying the boundary condition (9.2). As was mentioned in Section 9, it can be assumed in (5.2) that $A_1 = 1$ and the system reduces in a known manner to a harmonic equation, the solution of which can be written in the form of a series

$$S_{23} = \sum a_n \exp\left[-\frac{2n-1}{2}\pi\eta_1\right]\cos\frac{2n-1}{2}\pi\zeta \tag{10.2}$$

The series (10.2) satisfies, term-by-term, both the face requirements $S_{23} = 0$ when $\zeta = \pm 1$ and the decay conditions, and the edge value S_{12} is expressed in the form of a trigonometric series

$$S_{12}\big|_{\eta_1=0} = \sum a_n \sin\frac{2n-1}{2}\pi\zeta \tag{10.3}$$

In this series, the numerical coefficients a_n are defined in an obvious way for a specified right-hand side of equality (10.3). Consequently, the problem of constructing the antiplane boundary layer subject to the boundary condition

$$S_{12}\big|_{\alpha_1=0} = f(\zeta)$$

is solved in an elementary manner for an extremely wide class of odd functions $f(\zeta)$.

In particular, when the boundary condition has the form (9.2), that is, when it is an issue of the standard problem in Section 9, only the first term of the series can be retained on the right-hand side of equality (10.2) with a sufficient accuracy. However, when $n = 1$, the harmonic equation for S_{23} is practically identical with Eq. (10.1) if the term in square brackets is discarded in the second equation and the variable ζ is eliminated from the first equation using the substitution $S_{23} = s_n(\eta_1)\sin((2n-1)/2)\pi\zeta$ (we recall that, in (10.1) and in [12], h is to be understood not as the half-thickness of the plate but its overall thickness).

11. It follows from what has been described above that asymptotic analysis confirms the possibility of using the shear theory of bending for the complete (including the boundary zones) approximate analysis of the SSS of a plate only when, firstly, its edge is free and, secondly, the conditions of the problem envisage a fairly simple law for the distribution of the stresses σ_{12}, which are transmitted to the edge of the plate, throughout the thickness (the second requirement is certainly satisfied when no external forces are applied to the edge of the shell). At the same time, the asymptotic and shear theories turn out to be inadequate not only from the point of view of the final result but also from the point of view of the simplicity of the mathematical apparatus.

The assertion which has been put forward rests largely on the fact that the asymptotic relations

$$(\mathrm{SSS})_{\mathrm{int}} \sim \eta^{\alpha}(\mathrm{SSS})^{a}_{\mathrm{bound}} >> \eta^{\beta}(\mathrm{SSS})^{b}_{\mathrm{bound}}$$

hold; these follow from formula (6.3) for the exponents α and β.

If the edge of the plate is clamped or supported by a hinge, then the exponents α and β will be determined by formula (7.3) instead of (6.3)

$$(\mathrm{SSS})_{\mathrm{int}} \sim \eta^{\beta}(\mathrm{SSS})^{b}_{\mathrm{bound}} >> \eta^{\alpha}(\mathrm{SSS})^{b}_{\mathrm{bound}}$$

They show that, in such cases, the investigation of boundary elastic phenomena in a bending plate using shear theories has no meaning. In the best case, it is only possible with such an approach to construct solely the asymptotically secondary term $\eta^{\alpha}(\mathrm{SSS})^{a}_{\mathrm{bound}}$ in the structural formula (5.5).

The asymptotic approach also holds good in the case of the bending of plates with a rigid or hinged edge. In these cases, it only becomes somewhat more complex. Instead of a harmonic problem of an antiplane boundary layer, it is necessary to solve a biharmonic problem of a plane boundary layer and, consequently, instead of the method of trigonometric series, it is necessary to use Papkovich's method, for example.

For shells which do not degenerate into a plate, at the present time it is impossible to consider the construction of shear theories as being completed. Papers [4, 15] which deal with this problem have revealed the great unwieldiness of the corresponding formulae and equations, and the question as to the correctness of the use of such equations for a boundary calculation has not been raised until now. At the same time, no new treatments are required for the extension of the technique described here of separate internal and boundary calculations to shells. The Kirchhoff–Love theory holds good for an approximate internal calculation while, as previously, for the boundary calculation it will be necessary to solve the well-studied antiplane and plane problems of the theory of elasticity. The decaying solutions, for which the curvature of the shell is a secondary factor, will be subject to construction. Hence, the specific nature of the boundary calculations of shells, which do not degenerate into a plate, will lie only in the fact that the number of cases requiring the treatment of plane boundary layers is increased. The need to overcome the complexity of the shear theory of shells becomes superfluous.

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